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ON BOUNDARIES OF THE GROWTH REGION OF MULTIDIMENSIONAL PERTURBATIONS OF UNSTABLE STATES

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When the development of instability in the coordinate-independent stationary states is investigated, the determination of the boundary of the expanding region within which the perturbations grow, initially specified in a bounded region is of great importance. In particular, the knowledge of these boundaries makes it possible to decide whether the instability is absolute or convective [1, 2]. Below it will be shown that a boundary of the region occupied by growing perturbations can be obtained, in the non-onedimensional case, in the form of an envelope of the straight lines or planes bounding the region in which the growth of the one-dimensional perturbations takes place.

We shall limit ourselves, for convenience, to investigating the two-dimensional perturbations (the three-dimensional perturbations are investigated in the same manner). As we know, a perturbation which is initially localized and represented by the Fourier integral [3]

$$u(x, y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[ik_1x + ik_2y - i\omega(k_1, k_2)t]\, dk_1 dk_2$$

can be estimated along the rays $x = Ut,\ y = Vt$ as $t \to \infty$, using the method of steepest descent [3] in accordance with the formulas

$$\exp[t \operatorname{Im} \omega'(U, V) - \ln t] \tag{1}$$

$$\omega' = \omega(k_1, k_2) - k_1U - k_2V \tag{2}$$

$$\partial\omega / \partial k_1 = U, \quad \partial\omega / \partial k_2 = V \tag{3}$$

where $\omega = \omega(k_1, k_2)$ represents the dispersion equation of the perturbations. We find the function $\omega' = \omega'(U, V)$ by obtaining the values of k_1 and k_2 from (3), the latter defining the points of steepest descent of the function $\omega'(k_1, k_2, U, V)$ on the complex planes k_1 and k_2, and substituting these values into (2).

The curve defined by the equation

$$\operatorname{Im} \omega'(U, V) = 0 \tag{4}$$

separates the domain of values of U and V at which the perturbations grow along the ray $x = Ut,\ y = Vt$, from the domain corresponding to the decay of the perturbations.

Let us consider the one-dimensional perturbations corresponding to the same dispersion equation and such, that a coordinate system can be chosen so that $\operatorname{Im} k_2 = 0$.

The perturbations in this coordinate system will not grow along the y-axis, and we shall regard the real quantity k_2 as a parameter. The asymptotic behavior of the one-dimensional perturbations along the ray $x = Ut$ as $t \to \infty$ will have the form

$$\exp\left[t \operatorname{Im} \omega''(k_2, U) - {}^1/_2 \ln t\right] \tag{5}$$

$$\omega'' = \omega(k_1, k_2) - k_1 U, \quad \partial\omega/\partial k_1 = U \tag{6}$$

The boundaries of the domain on the U-axis corresponding to the growing perturbations are determined by the values of U extremal in k_2 and satisfying the equation

$$\operatorname{Im} \omega''(k_2, U) = 0 \tag{7}$$

Let the quantity U assume, at certain $k_2 = k_{20}$, its extremal value U_0. We shall show that a point can be found on the straight line $U = U_0$ lying in the U, V-plane, which belongs to the boundary curve (4). When $k_2 = k_{20}$ and $U = U_0$, we have

$$(\partial \operatorname{Im} \omega''/\partial k_2)_U = 0 \tag{8}$$

According to (6) we have

$$\left(\frac{\partial\omega''}{\partial k_2}\right)_U = \left(\frac{\partial\omega}{\partial k_1}\right)\left(\frac{\partial k_1}{\partial k_2}\right)_U + \left(\frac{\partial\omega}{\partial k_2}\right)_{k_1} - U\left(\frac{\partial k_1}{\partial k_2}\right) = \left(\frac{\partial\omega}{\partial k_2}\right)_{k_1} \tag{9}$$

From (8) and (9) it follows that a V_0 exists such that

$$(\partial\omega/\partial k_2)_{k_1} = V_0, \quad \operatorname{Im} V_0 = 0 \tag{10}$$

The fact that k_2 and V_0 are real implies, together with (7), that the relation (4) holds for the values U_0 and V_0, i.e. that the latter point belongs to the boundary curve.

We shall now show that the straight line $U = U_0$ touches the boundary curve (4) at the point U_0, V_0. Indeed, the derivative of $\operatorname{Im} \omega'$ along the direction of V will vanish only under the condition that the curve (4) has a vertical tangent. Using the relations (2) and (3), we obtain

$$\left(\frac{\partial \operatorname{Im} \omega'}{\partial V}\right)_U = \operatorname{Im}\left[\left(\frac{\partial\omega}{\partial k_1}\right)_{k_2}\left(\frac{\partial k_1}{\partial V}\right)_U + \left(\frac{\partial\omega}{\partial k_2}\right)_{k_1}\left(\frac{\partial k_2}{\partial V}\right)_U - k_2 - U \times \left(\frac{\partial k_1}{\partial V}\right)_U - V\left(\frac{\partial k_2}{\partial V}\right)U\right] = -\operatorname{Im} k_2$$

Considering now the straight lines of all possible orientations bounding the regions of growth of the one-dimensional perturbations, we obtain (4) as their envelope.

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